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Generic 2 by 2 Matrices and Periodic Resolutions

MARK RAMRAS

Northeastern University, Boston, Massachusetts 02115, and Rider College, Lawrenceville, New Jersey 08648

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INTRODUCTION

The purpose of this paper is to provide some new examples of periodic resolutions of period 2, by means of 2 by 2 generic matrices and 2 by 2 generic symmetric matrices.

Let (R, m) be a commutative noetherian local ring and let

$$\mathbb{F}: \cdots \to F_{i+1} \xrightarrow{d_i} F_i \xrightarrow{d_{i-1}} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_0} F_0$$

be an exact sequence of finitely generated free R-modules. We regard this as a free resolution of Coker d_0, and say that the resolution is *periodic of period n* if for all $i \geqslant 0, F_{i+n} = F_i$ and $d_{i+n} = d_i$. The resolution is *minimal* if $\operatorname{Im} d_i \subset mF_i$ for all $i \geqslant 0$.

Over commutatives rings, the only known periodic resolutions have period 2. Eisenbud [2, Theorem 4.1] proves that when R is a complete intersection, if $\mathbb{F}$ is a minimal free resolution and $\{\operatorname{rank} F_i\}$ is bounded, then $\mathbb{F}$ becomes periodic of period 2 after at most $1 + \dim R$ steps. He also notes that, in general, if a resolution has period 2 then $\{\operatorname{rank} F_i\}$ is constant. Thus resolutions of period 2 are in 1–1 correspondence with pairs (X, Y) of n by n matrices such that $\operatorname{Ker} X = \operatorname{Im} Y$ and $\operatorname{Ker} Y = \operatorname{Im} X$. Eisenbud also gives a construction of resolutions of period 2 which is general enough to account for every periodic minimal resolution over a complete intersection [2, Theorem 5.2]. The construction is this: let $x \in m$ be a non-zero-divisor on R, and let X and Y be n by n matrices such that $XY = xI_n = YX$, where I_n is the n by n identity matrix. Let "–" denote reduction modulo xR. Then

$$\cdots \to \bar{R}^n \xrightarrow{\bar{X}} \bar{R}^n \xrightarrow{\bar{Y}} \bar{R}^n \xrightarrow{\bar{X}} \bar{R}^n$$

is exact, and thus a periodic $\bar{R}$-free resolution of period 2.

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In the introduction in [2], Eisenbud mentions that all the known examples of periodic resolutions of period 2 over local rings arise from his construction. Thus we believe that our examples, which are constructed in a quite different, but very natural way, are of interest.

In Section 1 we give some results about R-sequences which we make frequent use of later. Two of them require that R contain a field K, and so that assumption is in force in Sections 2 and 3. Our first construction is given in Section 2, and the other, whose proof is fairly involved and depends upon the first, in Section 3.

The basic idea is simple. But first, some notation. For a matrix V, its transpose is denoted by V^T, and I_V will be the ideal generated by the entries of V. Now let X and Y be a *generic pair of* 2×2 *matrices*, i.e., we assume that their eight entries, taken together, form an R-sequence. Denoting by "$-$" reduction modulo I_{XY}, we clearly have a complex $\bar{R}^2 \xrightarrow{\bar{Y}} \bar{R}^2 \xrightarrow{\bar{X}} \bar{R}^2$, i.e., $\bar{X}\bar{Y} = 0$. In fact we show (Corollary 2.2) that $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$. Of course, this does not yield a periodic resolution of period 2 since $\bar{Y}\bar{X} \neq 0$. We get around this by proving the same result for a generic pair (X, Y) of *symmetric* 2×2 matrices, i.e., a pair whose six *distinct* entries form an R-sequences. For then $I_{YX} = I_{(YX)^T} = I_{X^TY^T} = I_{XY}$, and thus by symmetry, $\operatorname{Ker} \bar{Y} = \operatorname{Im} \bar{X}$, yielding a resolution of period 2.

We return to the generic pair of (nonsymmetric) 2×2 matrices in Section 3. Let $I = I_{XY} + I_{YX}$, and denote reduction modulo I by "$\sim$". Now $\tilde{X}\tilde{Y} = 0 = \tilde{Y}\tilde{X}$, and by symmetry, to prove that

$$\cdots \longrightarrow \tilde{R}^2 \xrightarrow{\tilde{X}} \tilde{R}^2 \xrightarrow{\tilde{Y}} \tilde{R}^2 \xrightarrow{\tilde{X}} \tilde{R}^2$$

is exact, it is only necessary to show that $\operatorname{Ker} \tilde{X} = \operatorname{Im} \tilde{Y}$. As we mentioned earlier, this depends upon the analogous result given in Section 2 for symmetric matrices. It is also necessary to know the associated primes of I in the ring of polynomials in the entries of X and Y with coefficients in the field K. We show (Lemma 3.2 and Theorem 3.5) that I is the intersection of three prime ideals: I_X, I_Y, and $(I, \det X, \det Y)$.

All rings in this paper are assumed to be commutative.

1. Properties of R-Sequences

Recall that if R is any commutative ring and M is any R-module, a sequence of elements $z_1, \ldots, z_n$ in R is an *M-sequence* if $(z_1, \ldots, z_n)M \neq M$, z_1 is regular (i.e., a non-zero-divisor) on M, and for $2 \leqslant i \leqslant n$, z_i is regular on $M/(z_1, \ldots, z_n)M$. Our main concern is with the case $M = R$.

We begin with a lemma from [6] which we shall use often.

Lemma 1.1 [6, Lemma 3.1]. *Let $z_1, \ldots, z_n$ be an R-sequence. Then $a_1 z_1 + \cdots +$*

$a_n z_n = 0 \Leftrightarrow [a_1, ..., a_n]^T = B[z_1, ..., z_n]^T$, *where* B *is an* $n \times n$ *alternating matrix, i.e., a skew-symmetric matrix with all* $B_{ii} = 0$.

Remark. We state explicitly that $B_{ii} = 0$ since in rings of characteristic 2 this is not implied by skew-symmetry.

An important and well-known feature of M-sequences is their permutability, given certain mild restrictions on R and M.

PROPOSITION 1.2. *Let* R *be a noetherian ring which is either:* (i) *local, with maximal ideal* m, *or* (ii) *graded. Let* $z_1, ..., z_n$ *be elements of* R, *which, in case* (i), *are in* m, *and in case* (ii), *are homogeneous of positive degree. Let* M *be an* R*-module which, if* R *is local, is finitely generated, and if* R *is graded, is also graded. Then if* $z_1, ..., z_n$ *is an* M*-sequence, so is any permutation of the* z*'s.*

Proof. The local case is [5, Theorem 119], and the graded case is [1, Corollary 2.9].

The next proposition allows one to deduce results about R-sequences from corresponding results for independent indeterminates over a field K, provided $R \supset K$, and R and the R-sequence satisfy the hypotheses of the preceding proposition. This technique will be demonstrated in the proof of the subsequent corollary and again in Sections 2 and 3.

PROPOSITION 1.3. *Let* R *and* $z_1, ..., z_n$ *satisfy the hypotheses of Proposition* 1.2. *Assume further that* R *contains a field* K. *Then* $z_1, ..., z_n$ *is an* R*-sequence* $\Leftrightarrow z_1, ..., z_n$ *are algebraically independent over* K *and* R *is a flat* $K[z_1, ..., z_n]$*-module.*

Proof. The local case is [3, Theorem] and the graded case is [7, Proposition 1.3].

COROLLARY 1.4. R *and* $z_1, ..., z_n$ *as above. If* $z_1, ..., z_n$ *is an* R*-sequence and* Q *is an ideal in* $K[z_1, ..., z_k]$, *where* $k < n$, *then* $z_{k+1}, ..., z_n$ *is an* R/QR*-sequence.*

Proof. Let $A = K[z_1, ..., z_n]$. Then R is A-flat, by Proposition 1.2. Furthermore, since the z's are algebraically independent over K, $A/QA \approx B[z_{k+1}, ..., z_n]$, where $B = K[z_1, ..., z_k]/Q$. Thus $z_{k+1}, ..., z_n$ is an A/QA-seqeuence. Since R is A-flat and $R/QR \approx A/QA \otimes_A R$, $z_{k+1}, ..., z_n$ is an R/QR-sequence.

2. For the remainder of this paper we assume that R is noetherian and contains a field K. Furthermore, R is either local with maximal ideal m, or else R is graded. X and Y will denote 2×2 matrices over R: $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$, where the x's and y's are in m if R is local, or if R is graded, are homogeneous with $\deg x_i$ and $\deg y_i$ positive constants. For any matrix V, I_V

will denote the ideal generated by the entries of V. We denote by "–" reduction modulo I_{XY}.

PROPOSITION 2.1. *Let $R = K[x_1, ..., x_4, w_1, ..., w_q]$ be a polynomial ring over the field K. Suppose that $y_i \in K[w_1, ..., w_q]$, $1 \leqslant i \leqslant 4$, are homogeneous polynomials of the same degree, and that either $y_1 \neq y_2$ or $y_3 \neq y_4$. Assume also that* $\det X$ *is regular on R/I_Y. Then over $\bar{R} = R/I_{XY}$,* $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$.

Proof. From the definition of I_{XY} it follows immediately that $\operatorname{Im} \bar{Y} \subset \operatorname{Ker} \bar{X}$. Now suppose $[\bar{z}_1, \bar{z}_2]^T \in \operatorname{Ker} \bar{X}$. Then, in particular, $x_1 z_1 + x_2 z_2 \in I_{XY}$. So

$$x_1 z_1 + x_2 z_2 = a_1(x_1 y_1 + x_2 y_3) + a_2(x_1 y_2 + x_2 y_4) \\ + a_3(x_3 y_1 + x_4 y_3) + a_4(x_3 y_2 + x_4 y_4).$$

Thus $x_1(z_1 - a_1 y_1 - a_2 y_2) + x_2(z_2 - a_1 y_3 - a_2 y_4) \in Q$, where $Q = (x_3 y_1 + x_4 y_3, x_3 y_2 + x_4 y_4)R$. We shall show that x_1, x_2 is an R/Q-sequence. First note that by our hypothesis on the y's, $x_3 y_1 + x_4 y_3$ and $x_3 y_2 + x_4 y_4$ are homogeneous polynomials in R. Now $R/(x_1, x_2) \approx K[x_3, x_4, w_1, ..., w_q]$ which is a UFD. Thus if $x_3 y_1 + x_4 y_3$, $x_3 y_2 + x_4 y_4$ were not an $R/(x_1, x_2)$-sequence, then these two elements of $K[w_1, ..., w_q][x_3, x_4]$ of degree 1 would have a common factor of degree 1, and hence they would be equal. But x_3 and x_4 are relatively prime and so $y_1 - y_2$ would be a multiple of x_4 and $y_3 - y_4$ a multiple of x_3. Since $y_i \in K[w_1, ..., w_q]$, both $y_1 - y_2$ and $y_3 - y_4$ would be 0, contradicting our assumption. Thus $x_3 y_1 + x_4 y_3$, $x_3 y_2 + x_4 y_4$ is an $R/(x_1, x_2)$-sequence. But x_1, x_2 is clearly an R-sequence, so $x_1, x_2, x_3 y_1 + x_4 y_3, x_3 y_2 + x_4 y_4$ is an R-sequence. Since the members of this sequence are all homogeneous polynomials of positive degree, the permuted sequence $x_3 y_1 + x_4 y_3$, $x_3 y_2 + x_4 y_4$, x_1, x_2 is an R-sequence (Proposition 1.2). Thus x_1, x_2 is an R/Q-sequence, as claimed. Therefore, for some $r \in R$,

$$z_1 - a_1 y_1 - a_2 y_2 \equiv r x_2 \pmod{Q},$$
$$z_2 - a_1 y_3 - a_2 y_4 \equiv -r x_1 \pmod{Q}.$$

Hence $[z_1, z_2]^T \equiv a_1[y_1, y_3]^T + a_2[y_2, y_4]^T + r[x_2, -x_1]^T \pmod{I_{XY}}$ since $Q \subset I_{XY}$. Now $[y_1, y_3]^T$ and $[y_2, y_4]^T$ are in $\operatorname{Im} Y$. Since $x_3 z_1 + x_4 z_2 \in I_{XY}$ by hypothesis, it follows that $r(x_2 x_3 - x_1 x_4) \in I_{XY}$, i.e., $r \det X \in I_{XY}$. But $I_{XY} \subset I_Y$, and by hypothesis, $\det X$ is regular on R/I_Y, so $r \in I_Y$. Let $r = \sum_{i=1}^4 b_i y_i$. We have

$$\bar{y}_1[\bar{x}_2, -\bar{x}_1] = \bar{x}_2[\bar{y}_1, \bar{y}_3]$$
$$\bar{y}_2[\bar{x}_2, -\bar{x}_1] = \bar{x}_2[\bar{y}_2, \bar{y}_4],$$
$$\bar{y}_3[\bar{x}_2, -\bar{x}_1] = -\bar{x}_1[\bar{y}_1, \bar{y}_3],$$
$$\bar{y}_4[\bar{x}_2, -\bar{x}_1] = -\bar{x}_1[\bar{y}_2, \bar{y}_4].$$

Hence $\bar{r}[\bar{x}_2, -\bar{x}_1]^T = \sum_{i=1}^4 \bar{b}_i \bar{y}_i[\bar{x}_2, -\bar{x}_1]^T \in \operatorname{Im} \bar{Y}$. Thus $[\bar{z}_1, \bar{z}_2]^T \in \operatorname{Im} \bar{Y}$, and so $\operatorname{Ker} \bar{X} \subset \operatorname{Im} \bar{Y}$, completing the proof.

COROLLARY 2.2. *Suppose $x_1, \ldots, x_4, y_1, \ldots, y_4$ is an R-sequence (i.e., X and Y are a "generic pair of* 2×2 *matrices"). Then* $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$.

Proof. By Proposition 1.3, R is A-flat, where A is the polynomial ring $K[x_1, \ldots, x_4, y_1, \ldots, y_4]$. Since $A/I_Y \approx K[x_1, \ldots, x_4]$, $\det X$ is regular on A/I_Y. The other hypotheses of Proposition 2.1 are obviously satisfied by A, and so the sequence $\bar{A}^2 \to^{\bar{Y}} \bar{A}^2 \to^{\bar{X}} \bar{A}^2$ is exact. Tensoring with the flat A-module R, we obtain the desired exact sequence $\bar{R}^2 \to^{\bar{Y}} \bar{R}^2 \to^{\bar{X}} \bar{R}^2$.

Essentially the same proof yields:

COROLLARY 2.3. *Suppose $x_1, \ldots, x_4, y_1, y_2, y_4$ is an R-sequence and $y_3 = y_2$ (so $Y = Y^T$). Then* $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$.

Next we consider the case where X is symmetric.

PROPOSITION 2.4. *Let $X = X^T$. Suppose x_1, x_2, x_4, y_3, y_4 is an R-sequence and* $\det X$ *is regular on R/I_Y. Then* $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$.

Proof. Let $[\bar{z}_1, \bar{z}_2]^T \in \operatorname{Ker} \bar{X}$. Then $\exists a_i \in R$, $1 \leqslant i \leqslant 4$, such that

$$\begin{aligned} x_1 z_1 + x_2 z_2 = {} & a_1(x_1 y_1 + x_2 y_3) + a_2(x_1 y_2 + x_2 y_4) \\ & + a_3(x_2 y_1 + x_4 y_3) + a_4(x_2 y_2 + x_4 y_4). \end{aligned}$$

Thus $x_1(z_1 - a_1 y_1 - a_2 y_2) + x_2(z_2 - a_1 y_3 - a_2 y_4 - a_3 y_1 - a_4 y_2) = x_4(a_3 y_3 + a_4 y_4)$. Since x_1, x_2, x_4 is an R-sequence, it follows from Lemma 1.1 that

$$\begin{aligned} z_1 &= a_1 y_1 + a_2 y_2 + b_{12} x_2 + b_{13} x_4, \\ z_2 &= a_1 y_3 + a_2 y_4 - b_{12} x_1 + b_{23} x_4 + a_3 y_1 + a_4 y_2, \end{aligned}$$

and

$$a_3 y_3 + a_4 y_4 = b_{13} x_1 + b_{23} x_2 \tag{*}$$

for some $b_{ij} \in R$, $1 \leqslant i, j \leqslant 3$. Hence

$$\begin{aligned} [z_1, z_2] = {} & a_1[y_1, y_3] + a_2[y_2, y_4] + b_{12}[x_2, -x_1] \\ & + x_4[b_{13}, b_{23}] + [0, a_3 y_1 + a_4 y_2]. \end{aligned} \tag{**}$$

By Proposition 1.2, x_1, x_2, y_3, y_4 is an R-sequence. Applying Lemma 1.1 to (*), there is a 4×4 alternating matrix $[c_{ij}]$ such that

$$[b_{13}, b_{23}, -a_3, -a_4] = [x_1, x_2, y_3, y_4][c_{ij}].$$

Thus $x_4[b_{13}, b_{23}] + [0, a_3 y_1 + a_4 y_2] = c_{21} x_4[x_2, -x_1] + c_{31}[x_4 y_3, x_1 y_1] + c_{41}[x_4 y_4, x_1 y_2] + c_{32}[0, x_2 y_1 + x_4 y_3] + c_{42}[0, x_2 y_2 + x_4 y_4] + c_{34}[0, \det Y]$.

Substituting into (**) the right side of this equation for the left, and reducing modulo I_{XY}:

$$[\bar{z}_1, \bar{z}_2] = (\bar{a}_1 - \bar{c}_{34}\bar{y}_2 + \bar{c}_{13}\bar{x}_2)[\bar{y}_1, \bar{y}_3] + (\bar{a}_2 + \bar{c}_{34}\bar{y}_1 + \bar{c}_{14}\bar{x}_2)[\bar{y}_2, \bar{y}_4] \\ + (\bar{b}_{12} + \bar{c}_{21}\bar{x}_4)[\bar{x}_2, -\bar{x}_1].$$

By hypothesis, $\bar{x}_2\bar{z}_1 + \bar{x}_4\bar{z}_2 = 0$. Furthermore $[\bar{y}_1, \bar{y}_3]^T$ and $[\bar{y}_2, \bar{y}_4]^T$ are in $\operatorname{Im} \bar{Y} \subset \operatorname{Ker} \bar{X}$. Therefore $(b_{12} + c_{21}x_4) \det X \in I_{XY} \subset I_Y$. Since $\det X$ is assumed to be regular on R/I_Y, $b_{12} + c_{21}x_4 = \sum_{i=1}^{4} d_i y_i$. By the same argument as the one given at the end of the proof of Proposition 2.1, it follows that $(\bar{b}_{12} + \bar{c}_{21}\bar{x}_4)[\bar{x}_2, -\bar{x}_1]^T \in \operatorname{Im} \bar{Y}$. Hence $[\bar{z}_1, \bar{z}_2]^T \in \operatorname{Im} \bar{Y}$. Thus $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$.

COROLLARY 2.5. *Let $X = X^T$. Suppose $x_1, x_2, x_4, y_1, y_2, y_3, y_4$ is an R-sequence. Then* $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$.

Proof. We need only show that $\det X$ is regular on R/I_Y, and this follows, since R is A-flat, where $A = K[x_1, x_2, x_4, y_1, \ldots, y_4]$, from the regularity of $\det X$ on $A/(y_1, \ldots, y_4)$.

Combining Corollaries 2.3 and 2.5 we obtain:

COROLLARY 2.6. *Let $Y = Y^T$. Suppose $x_1, x_2, x_3, x_4, y_1, y_2, y_4$ is an R-sequence. Then* $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$ *and* $\operatorname{Ker} \bar{Y} = \operatorname{Im} \bar{X}^T$.

Proof. The first statement is just Corollary 2.3. For the second, substitute Y for X and X^T for Y in Corollary 2.5 and note that $I_{XY} = I_{(XY)^T} = I_{Y^TX^T} = I_{YX^T}$.

We come now to the last and main result of this section, which yields our first example of a periodic resolution of period 2.

COROLLARY 2.7. *Let $X = X^T$ and $Y = Y^T$. Suppose $x_1, x_2, x_4, y_1, y_2, y_4$ is an R-sequence. Then* $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$ *and (by symmetry)* $\operatorname{Ker} \bar{Y} = \operatorname{Ker} \bar{X}$.

Proof. This follows from Proposition 2.4 in the same way that Corollary 2.5 did.

3. We have seen (Corollary 2.2) that if (X, Y) is a generic pair of 2×2 matrices, then, modulo the smallest ideal such that $XY \equiv 0$, $\operatorname{Ker} \bar{X} = \operatorname{Im} \bar{Y}$. This suggests that to obtain a resolution of period 2 we consider the ideal $I = I_{XY} + I_{YX}$, which is the smallest ideal modulo which both $XY \equiv 0$ and $YX \equiv 0$. Denote reduction modulo I by "$\sim$". The main result of this section is

COROLLARY 3.8. *Let $x_1, \ldots, x_4, y_1, \ldots, y_4$ be an R-sequence. Then* $\operatorname{Ker} \tilde{X} = \operatorname{Im} \tilde{Y}$ *and* $\operatorname{Ker} \tilde{Y} = \operatorname{Im} \tilde{X}$.

The proof is rather long, and depends on Corollary 2.7 and on establishing a primary decomposition for I in the case $R = K[x_1, \ldots, x_4, y_1, \ldots, y_4]$. We begin with a simple lemma which will be used toward the end of this section.

LEMMA 3.1. *Let A be any ring, $[a_1, a_2]^T \in A^2$, and X any 2×2 matrix. Let X^* be the classical adjoint of X. Suppose that $X[a_1, a_2]^T = 0$. Then $a_i(\operatorname{Im} X^*) \subset [a_1, a_2]^T A$, for $i = 1, 2$.*

Proof. $X^* = \left[\begin{smallmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{smallmatrix}\right]$, and by hypothesis, $x_2 a_2 = -x_1 a_1$ and $x_4 a_2 = -x_3 a_1$. Thus if $\{e_1, e_2\}$ is the standard basis of A^2,

$$a_1 X^* e_1 = [x_4 a_1, -x_3 a_1]^T = [x_4 a_1, x_4 a_2]^T = x_4[a_1, a_2]^T,$$
$$a_1 X^* e_2 = [-x_2 a_1, x_1 a_1]^T = [-x_2 a_1, -x_2 a_2]^T = -x_2[a_1, a_2]^T,$$

and similarly, we find $a_2 X^* e_i \in [a_1, a_2]^T A$ for $i = 1, 2$.

From now on we let $A = K[x_1, \ldots, x_4, y_1, \ldots, y_4]$, where the x's and y's are algebraically independent over K. Let $J = (I, \det X, \det Y)A$, where, as we stated earlier, $I = I_{XY} + I_{YX}$.

LEMMA 3.2. (i) *$J \cap I_X = (I, \det X)A$, and* (ii) *$J \cap I_X \cap I_Y = I$.*

Proof. (i) Let $r \in J \cap I_X$. Then since $r \in J$, $r = a + b(\det X) + c(\det Y)$, where $a \in I$. Since $(I, \det X) \subset I_X$ and $r \in I_X$, we have $c(\det Y) \in I_X$. But $I_X = (x_1, \ldots, x_4)$ is prime in A, $\det Y \notin I_X$, and by Cramer's rule, $I_X(\det Y) \subset I_{YX}$, so $r \in (I, \det X)$. Therefore $J \cap I_X \subset (I, \det X)$, and since the reverse inclusion is obvious, equality is established.

For (ii) we must show that $(I, \det X) \cap I_Y = I$. The proof is quite similar to that for (i) and so we omit it.

Once we have proved that J is prime, Lemma 3.2 (ii) will give the primary decomposition of I as the intersection of three primes. The first step is to localize at the multimplicative set $\{x_1^n \mid n \geqslant 0\}$, or equivalently, to adjoin x_1^{-1} to A.

LEMMA 3.3. *$JA[x_1^{-1}]$ is prime in $A[x_1^{-1}]$.*

Proof. It follows from Cramer's rule that $x_i(\det Y) \in I_{YX}$ and $y_i(\det X) \in I_{XY}$, $i = 1, \ldots, 4$. Furthermore, $y_i(\det X) = y_i(\det X^T) \in I_{X^T Y^T} = I_{(YX)^T} = I_{YX}$. In particular,

$$\begin{aligned}
x_1(x_3 y_1 + x_4 y_3) &= x_3(x_1 y_1 + x_2 y_3) + y_3(\det X),\\
x_1(x_3 y_2 + x_4 y_4) &= x_3(x_1 y_2 + x_2 y_4) + y_4(\det X),\\
x_1(x_2 y_1 + x_4 y_2) &= x_2(x_1 y_1 + x_3 y_2) + y_2(\det X),\\
x_1(x_2 y_3 + x_4 y_4) &= x_2(x_1 y_3 + x_3 y_4) + y_4(\det X),\\
x_1(\det Y) &= y_4(x_1 y_1 + x_2 y_3) - y_3(x_1 y_2 + x_2 y_4).
\end{aligned}$$

Hence $JA[x_1^{-1}]$ is generated by the following set:

$$\{x_1y_1 + x_2y_3, x_1y_2 + x_2y_4, x_1y_1 + x_3y_2, x_1y_3 + x_3y_4, \det X\}.$$

So $JA[x_1^{-1}] = (y_1 + x_2y_3x_1^{-1}, y_1 + x_3y_2x_1^{-1})A[x_1^{-1}] + L$, where $L = (y_2 + x_2y_4x_1^{-1}, y_3 + x_3y_4x_1^{-1}, x_4 - x_2x_3x_1^{-1})\, A[x_1^{-1}]$. Let $A' = A[x_1^{-1}]/L$, and $J' = JA[x_1^{-1}]/L$. Then

$$A' \approx K[x_1, x_2, x_3, y_1, y_4][x_1^{-1}].$$

$A[x_1^{-1}]/JA[x_1^{-1}] \approx A'/J' \approx K[x_1, x_2, x_3, y_1, y_4][x_1^{-1}]/(y_1 - x_2x_3y_4x_1^{-2}) \approx K[x_1, x_2, x_3, y_4][x_1^{-1}]$ which is clearly a domain. Hence $JA[x_1^{-1}]$ is prime.

To prove that J is prime in A it suffices to prove:

PROPOSITION 3.4. *x_1 is regular on A/J.*

Proof. Suppose $x_1 f \in J$. Then $x_1 f \in J \cap I_X = (I, \det X)$ (Lemma 3.2(i)). So

$$\begin{aligned} x_1 f = {} & a_1(x_1y_1 + x_2y_3) + a_2(x_1y_2 + x_2y_4) + a_3(x_3y_1 + x_4y_3) \\ & + a_4(x_3y_2 + x_4y_4) + b_1(x_1y_1 + x_3y_2) + b_2(x_2y_1 + x_4y_2) \\ & + b_3(x_1y_3 + x_3y_4) + b_4(x_2y_3 + x_3y_4) + \alpha(x_1x_4 - x_2x_3). \end{aligned}$$

Rearranging, we have:

$$\begin{aligned} & x_1(f - (a_1 + b_1)y_1 - a_2y_2 - b_3y_3 - \alpha x_4) \\ & \quad = x_2((a_1 + b_4)y_3 + a_2y_4 + b_2y_1 - \alpha x_3) + y_2((a_4 + b_1)x_3 + b_2x_4) \\ & \qquad + y_4((a_4 + b_4)x_4 + b_3x_3) + a_3(x_3y_1 + x_4y_3). \end{aligned} \tag{1}$$

Now let $Q = (x_3y_1 + x_4y_3)A$, and reduce (1) modulo Q:

$$\begin{aligned} & x_1(f - (a_1 + b_1)y_1 - a_2y_2 - b_3y_2 - b_3y_3 - \alpha x_4) \\ & \quad \equiv x_2((a_1 + b_4)y_3 + a_2y_4 + b_2y_1 - \alpha x_3) + y_2((a_4 + b_1)x_3 + b_2x_4) \\ & \qquad + y_4((a_4 + b_4)x_4 + b_3x_3) \qquad (\mathrm{mod}\ Q). \end{aligned} \tag{2}$$

Now $x_3y_1 + x_4y_3$ is clearly regular on $A/(x_1, x_2, y_2, y_4) \approx K[x_3, x_4, y_1, y_3]$. Thus $x_1, x_2, y_2, y_4, x_3y_1 + x_4y_3$ is an A-sequence, and since these polynomials are all homogeneous, the permuted sequence $x_3y_1 + x_4y_3, x_1, x_2, y_2, y_4$ is also an A-sequence (Proposition 1.2). Hence by Lemma 1.1 there is a 4×4 alternating matrix $[r_{ij}]$ such that

$$\begin{bmatrix} -f + (a_1 + b_1)\, y_1 + a_2y_2 + b_3y_3 + \alpha x_4 \\ (a_1 + b_4)\, y_3 + a_2y_4 + b_2y_1 - \alpha x_3 \\ (a_4 + b_1)\, x_3 + b_2x_4 \\ (a_4 + b_4)\, x_4 + b_3x_3 \end{bmatrix} \equiv [r_{ij}] \begin{bmatrix} x_1 \\ x_2 \\ y_2 \\ y_4 \end{bmatrix} \qquad (\mathrm{mod}\ Q).$$

Since $Q \subset J$,

$$f \equiv (a_1 + b_1)y_1 + a_2y_2 + b_3y_3 + \alpha x_4 - r_{12}x_2 - r_{13}y_2 - r_{14}y_4 \pmod{J} \tag{3}$$

We also get:

$$r_{12}x_1 - r_{23}y_2 - r_{24}y_4 + (a_1 + b_4)y_3 + a_2y_4 + b_2y_1 - \alpha x_3 = s_1(x_3y_1 + x_4y_3), \tag{4}$$

$$r_{13}x_1 + r_{23}x_2 - r_{34}y_4 + (a_4 + b_1)x_3 + b_2x_4 = s_2(x_3y_1 + x_4y_3), \tag{5}$$

$$r_{14}x_1 + r_{24}x_2 + r_{34}y_2 + (a_4 + b_4)x_4 + b_3x_3 = s_3(x_3y_1 + x_4y_3). \tag{6}$$

Rearranging (4), (5), and (6) yields:

$$r_{12}x_1 - (\alpha + s_1y_1)x_3 + b_2y_1 + (a_1 + b_4 - s_1x_4)y_3 - r_{23}y_2 + (a_2 - r_{24})y_4 = 0, \tag{4'}$$

$$r_{13}x_1 + r_{23}x_2 + (a_4 + b_1 - s_2y_1)x_3 + (b_2 - s_2y_3)x_4 - r_{34}y_4 = 0, \tag{5'}$$

$$r_{14}x_1 + r_{24}x_2 + (b_3 - s_3y_1)x_3 + (a_4 + b_4 - s_3y_3)x_4 + r_{34}y_2 = 0. \tag{6'}$$

To solve this system, we begin with (6′). Since x_1, x_2, x_3, x_4, y_2 is an A-sequence, there is a 5×5 alternating matrix $[c_{ij}]$ such that

$$[r_{14}, r_{24}, b_3 - s_3y_1, a_4 + b_4 - s_3y_3, r_{34}]^T = [c_{ij}][x_1, x_2, x_3, x_4, y_2]^T. \tag{7}$$

Substituting the values thus obtained for r_{34} and a_4 into (5′) and rearranging, we have

$$\begin{aligned}(r_{13} - c_{14}x_3 + c_{15}y_4)x_1 + (r_{23} - c_{24}x_3 + c_{25}y_4)x_2 \\ + (b_1 - b_4 - c_{34}x_3 - s_2y_1 + s_3y_3 + c_{45}y_2 + c_{35}y_4)x_3 \\ + (b_2 - s_2y_3 + c_{45}y_4)x_4 = 0.\end{aligned} \tag{8}$$

Since x_1, x_2, x_3, x_4 is an A-sequence, there is a 4×4 alternating matrix $[d_{ij}]$ such that

$$\begin{aligned}\begin{bmatrix} r_{13} \\ r_{23} \\ b_1 - b_4 \\ b_2 \end{bmatrix} = x_3 \begin{bmatrix} c_{14} \\ c_{24} \\ c_{34} \\ 0 \end{bmatrix} - y_4 \begin{bmatrix} c_{15} \\ c_{25} \\ c_{35} \\ c_{45} \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 0 \\ -s_3 \\ s_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s_2y_1 - c_{45}y_2 \\ 0 \end{bmatrix} \\ + [d_{ij}][x_1, x_2, x_3, x_4]^T.\end{aligned} \tag{9}$$

Substituting into (4′) the values obtained from (9) for r_{23}, b_2, and b_4, and from (7) for r_{24}, we obtain, after simplifying and rearranging:

$$\begin{aligned}&x_1(r_{12} - d_{14}y_1 + d_{12}y_2 + d_{13}y_3 + c_{12}y_4) + x_2(-d_{24}y_1 + d_{23}y_3) \\ &\quad - x_3(\alpha + (d_{34} + s_1)y_1 + (c_{24} + d_{23})y_2 + c_{34}y_3 + c_{23}y_4) \\ &\quad - x_4(d_{24}y_2 + (d_{34} + s_1)y_3 + c_{24}y_4) + y_3(a_1 + b_1 + c_{45}y_2 + s_3y_3 + c_{35}y_4) \\ &\quad + y_4(a_2 - c_{45}y_1) = 0.\end{aligned} \tag{10}$$

Since $x_1, x_2, x_3, x_4, y_3, y_4$ is an A-sequence, there is a 6×6 alternating matrix k_{ij} such that

$$\begin{bmatrix} r_{12} - d_{14}y_1 + d_{12}y_2 + d_{13}y_3 + c_{12}y_4 \\ -d_{24}y_1 + d_{23}y_3 \\ \alpha + (d_{34} + s_1)\, y_1 + (c_{24} + d_{23})\, y_2 + c_{34}y_3 + c_{23}y_4 \\ d_{24}y_2 + (d_{34} + s_1)\, y_3 + c_{24}y_4 \\ a_1 + b_1 + c_{45}y_2 + s_3y_3 + c_{35}y_4 \\ a_2 - c_{45}y_1 \end{bmatrix} = [k_{ij}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_3 \\ y_4 \end{bmatrix}. \qquad (11)$$

In particular, looking at the second and fourth components of (11):

$$d_{24}y_1 + (k_{25} - d_{23})y_3 + k_{26}y_4 - k_{12}x_1 - k_{23}x_3 - k_{24}x_4 = 0, \qquad (12)$$

$$d_{24}y_2 + (d_{34} + s_1 - k_{45})y_3 + (c_{24} - k_{46})y_4 + k_{14}x_1 - k_{24}x_2 - k_{34}x_3 = 0. \qquad (13)$$

Using (7), (9), and (11) to substitute back into (3), we find, after simplifying:

$$\begin{aligned} -f \equiv x_4[(d_{34} + s_1 - k_{45})y_1 + (d_{23} + c_{24} - k_{46})y_2 - k_{14}x_2 + k_{34}x_4] \\ + x_2[k_{25}y_1 + k_{26}y_2 + k_{12}x_2 + k_{23}x_4] \qquad (\text{mod } J). \end{aligned} \qquad (14)$$

In order to evaluate the right side of (14) we shall first solve (12) and then (13).

Since $y_1, y_3, y_4, x_1, x_3, x_4$ is an A-sequence, there is a 6×6 alternating matrix $[p_{ij}]$ such that

$$[d_{24}, k_{25} - d_{23}, k_{26}, -k_{12}, -k_{23}, -k_{24}]^T = [p_{ij}][y_1, y_3, y_4, x_1, x_3, x_4]^T. \qquad (15)$$

Substituting the values obtained for d_{24} and k_{24} into (13) and rearranging, we have:

$$\begin{aligned} y_3(d_{34} + s_1 - k_{45} + p_{12}y_2 + p_{26}x_2) + y_4(c_{24} - k_{46} + p_{13}y_2 + p_{36}x_2) \\ + x_1(+k_{14} + p_{14}y_2 + p_{46}x_2) + x_3(-k_{34} + p_{15}y_2 + p_{56}x_2) \\ + p_{16}(x_2y_1 + x_4y_2) = 0. \end{aligned} \qquad (16)$$

Now $x_2y_1 + x_4y_2$ is regular on $A/(y_3, y_4, x_1, x_3) \approx K[x_2, x_4, y_1, y_2]$ and so $y_3, y_4, x_1, x_3, x_2y_1 + x_4y_2$ is an A-sequence. Since $(x_2y_1 + x_4y_2) \subset J$, there is a 4×4 alternating matrix $[q_{ij}]$ such that

$$\begin{bmatrix} d_{34} + s_1 - k_{45} \\ c_{24} - k_{46} \\ +k_{14} \\ -k_{34} \end{bmatrix} \equiv y_2 \begin{bmatrix} p_{12} \\ p_{13} \\ p_{14} \\ p_{15} \end{bmatrix} + x_2 \begin{bmatrix} p_{26} \\ p_{36} \\ p_{46} \\ p_{56} \end{bmatrix} + [q_{ij}] \begin{bmatrix} y_3 \\ y_4 \\ x_1 \\ x_3 \end{bmatrix} \qquad (\text{mod } J). \qquad (17)$$

If we now use (15) and (17) to substitute back into (14), we find that $f \equiv 0$ (mod J). Thus x_1 is regular on A/J.

Combining Lemma 3.3 and Proposition 3.4 we have:

THEOREM 3.5. *J is prime in A.*

Remark. $(I_{XY}, \det X, \det Y)$ is also prime in A. In fact, Huneke [4] has proved the following more general result: Let X be an $r \times n$ matrix of indeterminates and Y an $n \times s$ matrix of indeterminates where all the indeterminates considered together are algebraically independent. Then the ideal in $Z[x_{ij}, y_{jk}]$ (where $Z =$ the integers) generated by I_{XY} together with all $a + 1 \times a + 1$ minors of X, and all $b + 1 \times b + 1$ minors of Y, is prime if $a + b \leqslant n$.

Recall that for ideals B and C in a ring S, $B : C = \{s \in S \mid sC \subset B\}$.

LEMMA 3.6. $I : (y_2 - y_3) = (I, \det X)$.

Proof. Since $y_i \det X \in I$, $(I, \det X) \subset I : (y_2 - y_3)$. To prove the reverse inclusion it suffices to show that $y_2 - y_3$ is regular on $A/(I, \det X)$. But if it is not, then $y_2 - y_3$ must belong to some associated prime of $(I, \det X)$. By Lemma 3.2(i), $(I, \det X) = J \cap I_X$, and since J and I_X are both prime, they are precisely the associated primes of $(I, \det X)$. Clearly $y_2 - y_3$ belongs to neither one, and so we are done.

We come now to the main results of this section.

THEOREM 3.7. *Let $S = A/I$. Then* $\mathrm{Ker}(X \otimes 1_S) = \mathrm{Im}(Y \otimes 1_S)$ *and* $\mathrm{Ker}(Y \otimes 1_S) = \mathrm{Im}(X \otimes 1_S)$.

Proof. By symmetry we need only prove the first equality. Clearly $\mathrm{Im}(Y \otimes 1_S) \subset \mathrm{Ker}(X \otimes 1_S)$. Let $z_1 = y_2 - y_3$ and $z_2 = x_2 - x_3$. Let $T = A/(z_1, z_2)$. Then $X \otimes 1_T$ and $Y \otimes 1_T$ are a generic pair of *symmetric* 2×2 matrices over $T \approx K[x_1, x_2, x_4, y_1, y_2, y_4]$.

Now if U_1 and U_2 are symmetric then $(U_1U_2)^T = U_2U_1$ and so $I_{U_1U_2} = I_{U_2U_1}$. Thus $I_{(X \otimes 1_T)(Y \otimes 1_T)} = I_{(Y \otimes 1_T)(X \otimes 1_T)}$. By definition, $I = I_{XY} + I_{YX}$. It follows that $(I + (z_1, z_2))/(z_1, z_2) = I_{(X \otimes 1_T)(Y \otimes 1_T)}$. Hence

$$S/(\bar{z}_1, \bar{z}_2) = A/(I, z_1, z_2) \approx T/I_{(X \otimes 1_T)(Y \otimes 1_T)}.$$

Let $\hat{S} = S/(\bar{z}_1)$ and $\hat{\hat{S}} = S/(\bar{z}_1, \bar{z}_2)$. Then by Corollary 2.7, $\mathrm{Ker}\,\hat{\hat{X}} = \mathrm{Im}\,\hat{\hat{Y}}$. The idea of the proof now is to show that this implies $\mathrm{Ker}\,\hat{X} = \mathrm{Im}\,\hat{Y}$, which in turn implies $\mathrm{Ker}(X \otimes 1_S) = \mathrm{Im}(Y \otimes 1_S)$.

So let $u \in \mathrm{Ker}\,\hat{X}$. Then $u = \hat{v}$, for some $v \in S^2$. Thus $\hat{\hat{v}} \in \mathrm{Ker}\,\hat{\hat{X}} = \mathrm{Im}\,\hat{\hat{Y}}$, so $\hat{\hat{v}} = \hat{\hat{Y}}(\hat{\hat{w}})$ for some $w \in S^2$. Hence $\hat{v} - \hat{Y}(\hat{w}) = \hat{z}_2\hat{v}_0$ for some $v_0 \in S^2$. Thus

$$\hat{z}_2\hat{X}(\hat{v}_0) = \hat{X}(\hat{z}_2\hat{v}_0) = \underset{\substack{\| \\ 0}}{\hat{X}(\hat{v})} - \underset{\substack{\| \\ 0}}{\hat{X}\hat{Y}(\hat{w})} = 0.$$

We claim that $\operatorname{ann}_{\hat{S}}(\hat{z}_2) = (\hat{y}_1, \hat{y}_2, \hat{y}_4)$. First note that $\hat{S}/(\hat{y}_1, \hat{y}_2, \hat{y}_4) \approx A/(y_1, y_2, y_3, y_4) \approx K[x_1, x_2, x_3, x_4]$, and so $\hat{z}_2 = \hat{x}_2 - \hat{x}_3$ is regular on $\hat{S}/(\hat{y}_1, \hat{y}_2, \hat{y}_4)$. Therefore $\operatorname{ann}_{\hat{S}}(\hat{z}_2) \subset (\hat{y}_1, \hat{y}_2, \hat{y}_4)$. For the reverse inclusion:

$$\begin{aligned}
\hat{y}_1(\hat{x}_2 - \hat{x}_3) &= (\text{since } x_3 y_1 + x_4 y_3 \in I)\ \hat{x}_2 \hat{y}_1 + \hat{x}_4 \hat{y}_3 \\
&= \hat{x}_2 \hat{y}_1 + \hat{x}_4 \hat{y}_2 = 0 \qquad (\text{since } x_2 y_1 + x_4 y_2 \in I); \\
\hat{y}_2(\hat{x}_2 - \hat{x}_3) &= -\hat{x}_1 \hat{y}_1 - \hat{x}_3 \hat{y}_2 = 0 \qquad (\text{since } x_1 y_1 + x_2 y_2,\ x_1 y_1 + x_3 y_2 \in I); \\
\hat{y}_4(\hat{x}_2 - \hat{x}_3) &= -\hat{x}_1 \hat{y}_2 - \hat{x}_3 \hat{y}_4 = -\hat{x}_1 \hat{y}_3 - \hat{x}_3 \hat{y}_4 = 0.
\end{aligned}$$

Now since $\hat{z}_2 \hat{X}(\hat{v}_0) = 0$, it follows that $\hat{X}(\hat{v}_0) \in (\hat{y}_1, \hat{y}_2, \hat{y}_4)\hat{S}^2$. Let $\hat{S}^\# = \hat{S}/(\hat{y}_1, \hat{y}_2, \hat{y}_4)$. Then $\hat{X}^\#(\hat{v}_0{}^\#) = 0$. But $\hat{S}^\# = A/(y_1, y_2, y_3, y_4)$, and so $\det \hat{X}^\#$ is regular on $\hat{S}^\#$. Therefore $\hat{X}^\#$ is 1–1. Hence $\hat{v}_0{}^\# = 0$, i.e., $\hat{v}_0 \in (\hat{y}_1, \hat{y}_2, \hat{y}_4)\hat{S}^2$. But then $\hat{z}_2 \hat{v}_0 = 0$, and thus $\hat{v} = \hat{Y}(\hat{w}) \in \operatorname{Im} \hat{Y}$, proving that $\operatorname{Ker} \hat{X} = \operatorname{Im} \hat{Y}$.

To show that $\operatorname{Ker}(X \otimes 1_S) = \operatorname{Im}(Y \otimes 1_S)$, we shall show, by induction, that for all $k \geqslant 1$,

$$\operatorname{Ker}(X \otimes 1_S) \subset \operatorname{Im}(Y \otimes 1_S) + \bar{z}_1{}^k S^2.$$

Let $\alpha \in \operatorname{Ker}(X \otimes 1_S)$. Then $\hat{\alpha} \in \operatorname{Ker} \hat{X} = \operatorname{Im} \hat{Y}$, and so $\hat{\alpha} = \hat{Y}(\tilde{\beta}_1)$, $\tilde{\beta}_1 \in \hat{S}^2$. Thus $\alpha - (Y \otimes 1_S)(\beta_1) = \bar{z}_1 \alpha_1$, for some $\alpha_1 \in S^2$, which proves our assertion for $k = 1$. Now assume it for $k = n \geqslant 1$:

$$\bar{z}_1(X \otimes 1_S)(\alpha_1) = (X \otimes 1_S)(\alpha) - (X \otimes 1_S)(Y \otimes 1_S)(\beta_1) = 0.$$

Hence $(X \otimes 1_S)(\alpha_1) \in \operatorname{ann}_S(\bar{z}_1)S^2 = \det(X \otimes 1_S)S^2$ (by Lemma 3.6), and so $(X \otimes 1_S)(\alpha_1) = \det(X \otimes 1_S) \cdot \gamma$, for some $\gamma \in S^2$. Thus $(X \otimes 1_S)(\alpha_1) = (X \otimes 1_S)(X \otimes 1_S)^*(\gamma)$, where U^* denotes the classical adjoint of U. Therefore $\alpha_1 - (X \otimes 1_S)^*(\gamma) \in \operatorname{Ker}(X \otimes 1_S)$. By our induction hypothesis, $\alpha_1 - (X \otimes 1_S)^*(\gamma) = (Y \otimes 1_S)(\beta_n) + \bar{z}_1{}^n \alpha_n$, for some $\alpha_n, \beta_n \in S^2$. Thus

$$\begin{aligned}
\alpha &= (Y \otimes 1_S)(\beta_1) + \bar{z}_1 \alpha_1 \\
&= (Y \otimes 1_S)(\beta_1) + \bar{z}_1(X \otimes 1_S)^*(\gamma) + \bar{z}_1(Y \otimes 1_S)(\beta_n) + \bar{z}_1^{n+1} \alpha_n .
\end{aligned}$$

By Lemma 3.1, $\bar{z}_1(X \otimes 1_S)^*(\gamma) \in \operatorname{Im}(Y \otimes 1_S)$. Hence

$$\alpha \in \operatorname{Im}(Y \otimes 1_S) + \bar{z}_1^{n+1} S^2.$$

The inductive step is complete and so $\operatorname{Ker}(X \otimes 1_S) \subset \bigcap_k (\bar{z}_1{}^k S^2 + \operatorname{Im}(Y \otimes 1_S))$.

Now S is a graded ring, since I is a homogeneous ideal in the polynomial ring A. The entries of Y are homogeneous of degree 1, so that $\operatorname{Im}(Y \otimes 1_S)$ is a graded submodule of S^2. Hence every associated prime P of the graded S-module $S^2/\operatorname{Im}(Y \otimes 1_S)$ is homogeneous. Since $(\bar{z}_1)S$ is also homogeneous, $(\bar{z}_1)S + P \neq S$. Thus by [8, Chap. VIII, Sect. 4, Theorem 8], $\bigcap_k (\bar{z}_1{}^k S^2 + \operatorname{Im}(Y \otimes 1_S)) =$

$\operatorname{Im}(Y \otimes 1_S)$. Hence $\operatorname{Ker}(X \otimes 1_S) \subset \operatorname{Im}(Y \otimes 1_S)$, and since the reverse inclusion is obvious, the equality is proved.

Letting $\tilde{R} = R/I$, we have, by the same flatness argument used to deduce Corollary 2.2 from Proposition 2.1,

COROLLARY 3.8. *Let $x_1, \ldots, x_4, y_1, \ldots, y_4$ be an R-sequence. Then* $\operatorname{Ker} \tilde{X} = \operatorname{Im} \tilde{Y}$ and $\operatorname{Ker} \tilde{Y} = \operatorname{Im} \tilde{X}$.

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